Reanalysis for Structural Modifications in Boundary-Element Response and Design Sensitivity Analysis

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A reanalysis technique is presented for obtaining changes in response variables due to changes in design parameters with the boundary-element method. The approach is considered for static response analysis and design sensitivity analysis. Two iterative schemes, one based on total quantities and another based on incremental quantities, are developed. These schemes avoid the computationally expensive step of factorization of the coefficient matrix for the modified structure. Numerical results are obtained for analyses using 1) single-zone discretization, 2) multizone discretization, and 3) multizone discretization with condensation. The convergence trends for both iterative algorithms are presented for the response analysis and the design sensitivity analysis. A substantial economy in computer time is obtained using the present developments and is demonstrated here through CPU timings.

Introduction

THE process of design and optimization starting from the initial design. In most cases, the modifications from one step to the next are small and are determined based on the results of the previous analysis. The repeated analysis is time consuming and is therefore the object of concern. A central problem is to permit a modification of the structural analysis due to the small change in design in a time appreciably less than that required for a reanalysis.[1]

A number of researchers have shown a great deal of interest in developing efficient techniques for reanalysis of structures. A comprehensive survey of these efforts was given by Arora[2] for static response of modified structures and for frequency response of modified structures. A recent study of iterative methods for the sensitivity of displacement, stress, and eigenvalue and eigenvector was presented by Yoon and Belegunda.[3] Most of the work in this area has been presented in the context of the finite-element method. The development of reanalysis techniques for the boundary-element method have not been reported in the literature.

Although the boundary-element method (BEM) has been in existence for a few decades,[4] the design sensitivity analysis for shape optimization using BEM has been attempted only recently. This is evident from the fact that the papers for design sensitivity analysis using BEM (for example, for structural systems[5-11] and for heat-transfer systems[12,13] have appeared mainly in the last two years. Since the developments for reanalysis techniques have been closely tied to the developments for sensitivity analysis,[2] it is now timely to explore the reanalysis techniques with regards to BEM.

The present paper investigates a simple approach to obtain the static response and the design sensitivity information for a modified structure. This approach obviates the need to repeatedly factorize the left-hand-side coefficient matrix as the structure is continuously modified to evolve to an optimum shape. The variables for the modified structure are instead obtained through an iterative procedure based on the coefficient matrix of the original structure. The approach is first studied by modeling the object using a single-zone discretization. The convergence trend of the iterative algorithms is studied using numerical data. A rapid convergence of the technique is observed even for significant modification of the shape of the object. The approach is next studied by modeling the object using a multizone discretization. Two cases are studied in this approach: case 1 deals with the multizone analysis without condensation and case 2 deals with the multizone analysis with condensation of zones.[14,15] Case 2 is observed to converge more rapidly since it involves iterating for fewer variables. It is also significantly more economic in its requirement of computer resources since the procedure operates on matrices of smaller dimensions.

Problem Formulation

The boundary-element method based on Somigliana's identity leads to the matrix equations given as[4]

$$[F_1]\{u_1\} = [G_1]\{t_1\} + \{f_1\} \tag{1}$$

$[F]$ and $[G]$ are the boundary-element system matrices; $\{u\}$ and $\{t\}$ are the vectors of nodal displacements and tractions, respectively, and after application of the boundary conditions they denote the unknown and the known nodal variables, respectively; $\{f\}$ is a vector including the effect of various types of body forces; and the subscript 1 denotes the initial configuration of the system being analyzed.

After the solution for Eq. (1) has been obtained, the structure undergoes a modification as follows: 1) alteration of a portion of the configuration of the structure; and 2) change in the loading case considered. Often these changes are dictated by the shape-optimization algorithms when seeking an optimal configuration for the structure. These algorithms require the derivatives $\{u\},_L$ with respect to a design variable X_L. These are computed as the solution of the matrix equation obtained through, the implicit differentiation of Eq. (1) with respect to the design variable X_L as

$$[F_1]\{u_1\},_L = [G_1]\{t_1\},_L + \{g_1\} \tag{2}$$

where

$$\{g_1\} = [G_1],_L\{t_1\} + \{f_1\},_L - [F_1],_L\{u_1\}$$

The objective to be achieved is to compute the response $\{u_2\}$ and the sensitivity $\{u_2\},_L$ of the modified structure by means of a technique which requires lesser computational effort than if the structure were to be completely recalculated *ab initio*. Although $\{u_2\}$ simply provides the response of the modified

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structure, $\{u_2\},_L$ is computed to enable further modification of the structure, if necessary.

Reanalysis for Static Response Using the Single-Zone Model

For the modified system denoted by the subscript 2, the response equation corresponding to Eq. (1) can be written as

$$[F_2]\{u_2\} = [G_2]\{t_2\} + \{f_2\} \tag{3}$$

Equation (3) can be expressed as

$$[F_1]\{_2\} = \{R_a\} - [\Delta F]\{u_2\} \tag{4}$$

where

$$\{R_a\} = [F_1]\{u_1\} + [G_1]\{\Delta t\} + [\Delta G]\{t_1\} + [\Delta G]\{\Delta t\} + \{\Delta b\} \tag{5a}$$

$$[\Delta G] = [G_2 - G_1],\ \{\Delta t\} = \{t_2\} - \{t_1\},\qquad \{\Delta b\} = \{b_2 - b_1\} \tag{5b}$$

Furthermore, expressing $\{u_2\} = \{u_1\} + \{\Delta u\}$, Eq. (4) can be rewritten as

$$[F_1]\{\Delta u\} = \{R_b\} - [\Delta F]\{\Delta u\} \tag{6}$$

where

$$[\Delta F] = [F_2] - [F_1] \tag{7a}$$

$$\{R_b\} = [G]_1\{\Delta t\} + [\Delta G]\{t_1\} + [\Delta G]\{\Delta t\} + [\Delta b] - [\Delta F]\{u_1\} \tag{7b}$$

Equation (1) has been used here in obtaining Eq. (7) in its present form. Equations (4) and (6) form the basis for a total iteration algorithm and an incremental iteration algorithm for reanalysis.

Total Iteration Algorithm

For this algorithm, Eq. (4) is written in an iterative form as

$$[F_1]\{u_2^i\} = \{R_a\} - [\Delta F]\{u_2^{i-1}\} \tag{8}$$

The steps involved in this iteration are described as follows:

Step 1: Set $i = 1$. Choose an error tolerance ϵ. $\{u_2^{i-1}\}$ is the initial guess for $\{u_2\}$, and may be selected as $\{u_1\}$ is the response of the modified structure and does not differ widely from $\{u_1\}$. Compute $\{R_a\}$.

Step 2: Compute $\{R_a\} - [\Delta F]\{u_2^{-1}\}$ as defined in Eq. (5).

Step 3: Solve for $\{u_2^i\}$ using Eq. (8).

Step 4: Check the convergence criterion, for example,

$$\|\{u_2^i\} - \{u_2^{i-1}\}\| / \|\{u_2^i\}\| \leq \epsilon \tag{9}$$

If satisfied, then step. Otherwise

Step 5: Set $\{u_2^{i-1}\} = \{u_2^i\}$, $i = i + 1$, and go to step 2.

It is noted that the repeated solution of Eq. (8) as required in step 3 involves the factorization of the matrix $[F_1]$. This factorization, however, is performed earlier while solving Eq. (1) and is saved and reused in each pass of step 3. This avoidance of refactorization of the system matrix results in the substantial economy of solution reported in this paper.

Incremental Iteration Algorithm

To develop this algorithm equation (6) is rewritten as

$$[F_1]\{\Delta u^i\} = \{R_b\} - [\Delta F]\{\Delta u^{i-1}\} \tag{10}$$

The following steps describe the incremental algorithm.

Step 1: Set $i = 1$. Choose an error tolerance ϵ. Set $\{\Delta u^o\} = \{o\}$. Compute $\{R_b\} - [\Delta F]\{\Delta u^{i-1}\}$ as shown on the right-hand side of Eq. (10).

Step 3: Solve for $\{\Delta u^i\}$ using the factorization of $[F_1]$ obtained previously for the solution of Eq. (1).

Step 4: Update

$$\{u_2\} = \{u_1\} + \{\Delta u^i\} \tag{11}$$

Step 5: Check for convergence

$$\|\{\Delta u\}^i\| / \|\{u_2\}^i\| \leq t \tag{12}$$

If satisfied, then stop. Otherwise

Step 6: Set $\{\Delta u^{i-1}\} = \{\Delta u^i\}$, $i = i + 1$, and go to step 2.

Again, the reuse of previous factorization of $[F_1]$ in step 3 results in the present approach being economic.

Reanalysis for Design Sensitivity Using The Single-Zone Model

Following the steps for the development of reanalysis for static response, the design sensitivity analysis equations for the modified system can be written similar to Eq. (2) for the original system as

$$[F_2]\{u_2\},_L = [G_2]\{t_2\},_L + [G_2],_L\{t_2\} + \{f_2\},_L - [F_2],_L\{u_2\} \tag{13}$$

Equation (13) can be rewritten in terms of matrices of the original system as

$$[F_1]\{u_2\},_L = \{R_a\}^- [\Delta F]\{u_{2,L}\} \tag{14}$$

where

$$\{R_a\} = [G_2]\{t_2,_L + [G_2],_L\{t_2\} + \{b_2\},_L - [F_2],_L\{u_2\} \tag{15}$$

Equation (14) can be expressed in its iteration form as

$$[F_1]\{u_2^i\},_L = \{R_a\} - [\Delta F]\{u_2^{i-1}\},_L \tag{16}$$

The iteration algorithm for the solution of Eq. (16) for $\{u_2\},_L$ follows the description of the total iteration algorithm described in the last section with $\{R_a\}$ as defined in Eq. (15).

For the incremental iteration algorithm, the sensitivity vector $\{u_2\},_L$ is expressed as

$$\{u_2\},_L = \{u_1\},_L + \{\Delta u_L\} \tag{17}$$

Substituting Eq. (17) into Eq. (14) leads to the incremental equation

$$[F_1]\{\Delta u_L\} = \{R_b\} - [\Delta F]\{\Delta u_L\} \tag{18}$$

where

$$\{R_b\} = [G_2],_L\{t_2\} + [F_2],_L\{u_2\} - [F_2]\{u_1\},_L \tag{19}$$

The incremental iteration algorithm for sensitivity analysis can now be developed using Eq. (18) in a fashion similar to the incremental iteration algorithm for static response described in the last section. It is seen from Eqs. (16) and (18) that both of the algorithms for reanalysis for design sensitivity need the factorization of $[F_1]$ which is obtained during the solution of Eq. (1). Hence, no additional matrix factorizations are required for the design sensitivity reanalysis.

Reduced Reanalysis with Multizone BEM Using Condensation

The advantages of BEM analysis using multizone discretization include better-conditioned and sparse blocked system matrices. Numerical data for the computational advantages for such a discretization was given in Ref. 14. A technique for the discretization of objects using multizones and condensation of arbitrarily selected zones to reduce the dimensions of the system matrix being factorized to obtain the solution has been given for the static response analysis[14] and for the design sensitivity analysis.[15]

The multizone BEM discretization is especially attractive for the analysis of structural modification when only a portion of the geometric configuration is modified. In this case, the portion being modified is modeled using a separate zone. Then in obtaining the system matrices $[F_2]$ and $[G_2]$ for the modified structure, only the contributions corresponding to the zone that includes the portion of the configuration being modified need to be recomputed. The contributions from all other zones remain the same. Since the computation of system matrices for BEM involve computationally expensive numerical integration schemes, this feature of multizone analysis for shape modification leads to a substantial economy. This fact has been advantageously exploited in Refs. 14 and 15.

The condensation of system matrices for static analysis and design sensitivity analysis presented in Refs. 14 and 15, respectively, can be used in conjunction with the present reanalysis. This reanalysis procedure being iterative in nature and since the condensation procedure allows the size of the system to be reduced, the iterations are now done on these reduced matrices resulting in substantial economy. Following Refs. 14 and 15, Eq. (1) is partitioned as

$$\begin{bmatrix} [F_{MM}] & [F_{MC}] \\ [F_{CM}] & [F_{CC}] \end{bmatrix}_1 \begin{Bmatrix} \{u_M\} \\ \{u_C\} \end{Bmatrix}_1 = \begin{bmatrix} [G_{MM}] & [G_{MC}] \\ [G_{CM}] & [G_{CC}] \end{bmatrix}_1 \begin{Bmatrix} \{t_M\} \\ \{t_C\} \end{Bmatrix}_1 + \begin{Bmatrix} \{f_M\} \\ \{f_C\} \end{Bmatrix}_1 \tag{20}$$

where the subscripts M and C refer to the master and condensed degrees of freedom, respectively. This leads to the equation

$$[M]_1\{u_M\}_1 = [P]_1\{t_M\}_1 + \{\bar{f}_M\}_1 \tag{21}$$

where

$$\{\bar{f}_M\}_1 = [Q_A]_1\{t_c\}_1 + [Q_B]_2\{f_c\}_1 + \{f_M\}_1 \tag{22a}$$

$$[M]_1 = [F_{MM}]_1 - [F_{MC}]_1[F_{CC}]_1^{-1}[F_{CM}]_1 \tag{22b}$$

$$[P]_1 = [G_{MM}]_1 - [F_{MC}]_1[F_{CC}]_1^{-1}[G_{CM}]_1 \tag{22c}$$

$$[Q_A]_1 = [G_{MC}]_1 - [F_{MC}]_1[F_{CC}]_1^{-1}[G_{CC}]_1 \tag{22d}$$

$$[Q_B]_1 = -[F_{MC}]_1[F_{CC}]_1^{-1} \tag{22e}$$

It is observed that Eq. (21) has the same form as that for Eq. (1). Hence, both the total iterative and the incremental iterative algorithms developed for the single-zone model can also be applied to the multizone condensed model. Similar arguments hold for the multizone condensed design sensitivity analysis.

Additional Computational Aspects

The iteration algorithms described above need the quantities $[G_1],[t_1],[G_1]_{,L},\{t_1\}_{,L}$, etc. for the computation of the vectors $\{R_a\}\{R_b\}$, etc. given in Eqs. (5), (7), (15), and (19). Often times these quantities that constitute the right-hand side (RHS) of Eqs. (1) and (2) are not separately stored. Instead these RHS quantities are operated upon and assembled "on the fly." Thus, Eq. (1), for example, is written as

$$[F_1]\{u_1\} = \{y_1\}, \tag{23}$$

$$\{y_1\} = [G_1]\{t_1\} + \{f_1\} \tag{24}$$

The matrices $[G_1],\{t_1\}$, and $\{f_1\}$ are not separately assembled for the entire system. Instead the operation described in Eq. (24) is carried out at the element level and appropriately added to directly generate the vector $\{y_1\}$ for the entire system. The iteration algorithms can then be developed directly in terms of $\{y_1\}$. For example, for Eq. (23), the corresponding equation for total iteration algorithm is obtained as

$$[F_1]\{u_2^i\} = \{y_2\} - [\Delta F]\{u_2^{i-1}\} \tag{25}$$

The incremental iteration algorithm can similarly be developed. These arguments also hold for the total iteration and the incremental iteration algorithms for design sensitivities.

Some important remarks regarding reanalysis using multizone condensation discretization are now stated. Since for this case the size of the solution vector for which the iterations are performed is small, the iteration process will converge more rapidly compared to the case with uncondensed zones. In addition, it is noted that the most drastic changes in the response variables occur closest to that portion of the object where the modifications occur. Farther away from this changing portion the response variables do not change much. This fact can be taken advantage of by condensing the zone which contains the portion being modified in terms of a zone interface which lies farther away from the modifications. This will further enhance the convergence of the iteration process. The zone interface which contains the master degrees of freedom and for which the iterations are done could also be selected so that fewer nodes lie on this interface and hence a more economic solution can be obtained. These strategies described above were implemented in the computer program for the study of the numerical exmaples discussed below.

Numerical Results

A set of numerical examples was studied to evaluate the performance of the present developments. These examples were used to demonstrate that the boundary-element equations for static response and design sensitivity analysis are amenable to reanalysis techniques. These example also provide a measure of the order of convergence of the iterative reanalysis algorithms and the economy of computer resources obtained using these algorithms. The computations and the corresponding CPU times reported here were obtained on the RIDGE 3200 computer system.

Reanalysis for a Single-Zone Model

A square plate with a central hole subjected to a pressure of 120 N/mm^2 on its outer edges was studied. A quarter-symmetry portion of the plate was shown in Fig. 1. This example was studied earlier by WU[16] for shape optimization. They started with an initial design corresponding to Design A shown in Fig. 1 and obtained an optimal shape of a circle for the hole which corresponds to Design C in Fig. 1. During the optimization process, in going from Design A to Design C, many intermediate configurations are generated. Each of these configurations is analyzed for both static response and design sensitivities.

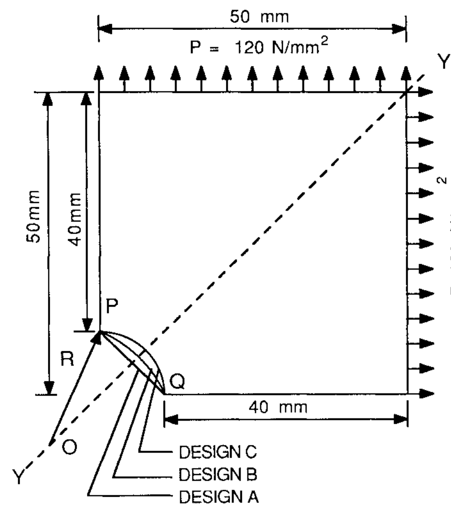

Fig. 1 A square plate with a central hole under uniform tension.

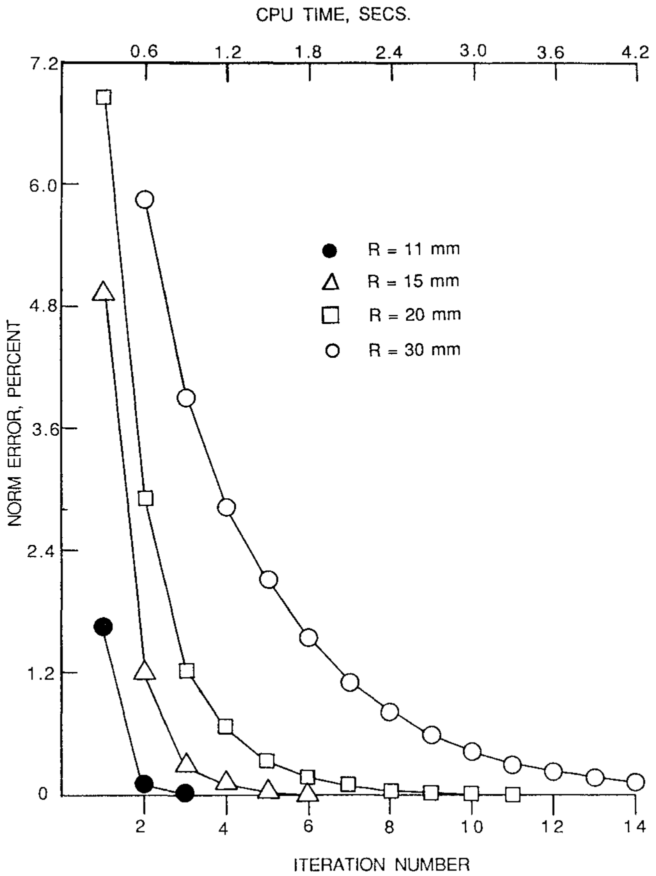

Fig. 2 Convergence of the total iteration algorithm with $\{u_2^o\}=\{u_1\}$ for the reanalysis of static responses.

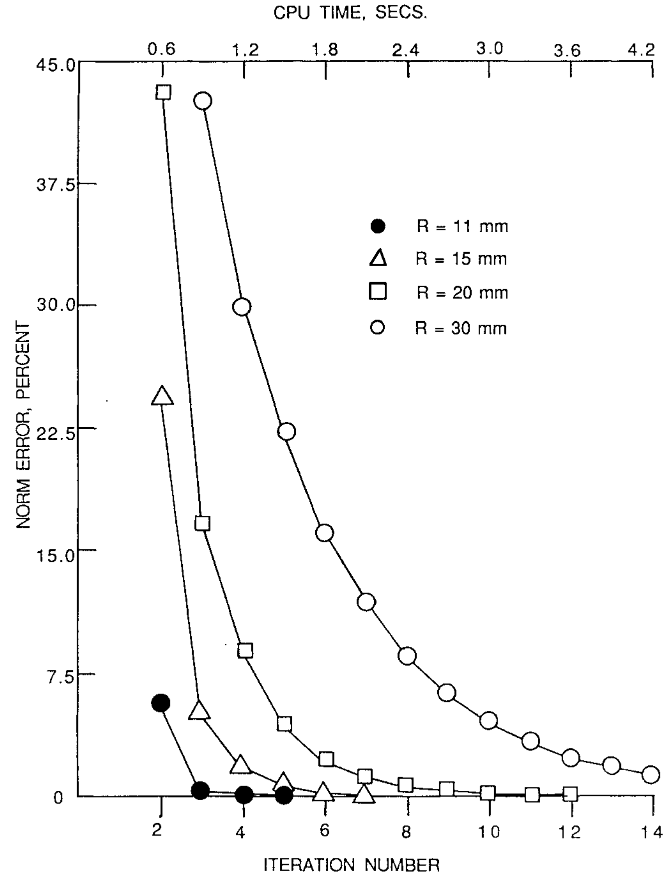

Fig. 4 Convergence of the incremental iteration algorithm for the reanalysis of static responses.

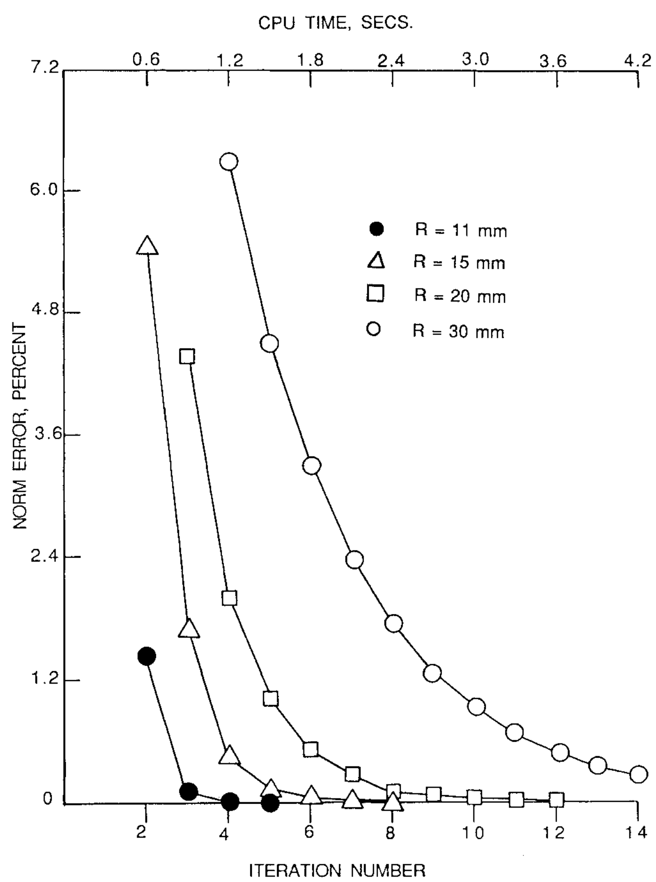

Fig. 3 Convergence of the total iteration algorithm with $\{u_2^o\}=\{O\}$ for the reanalysis of static responses.

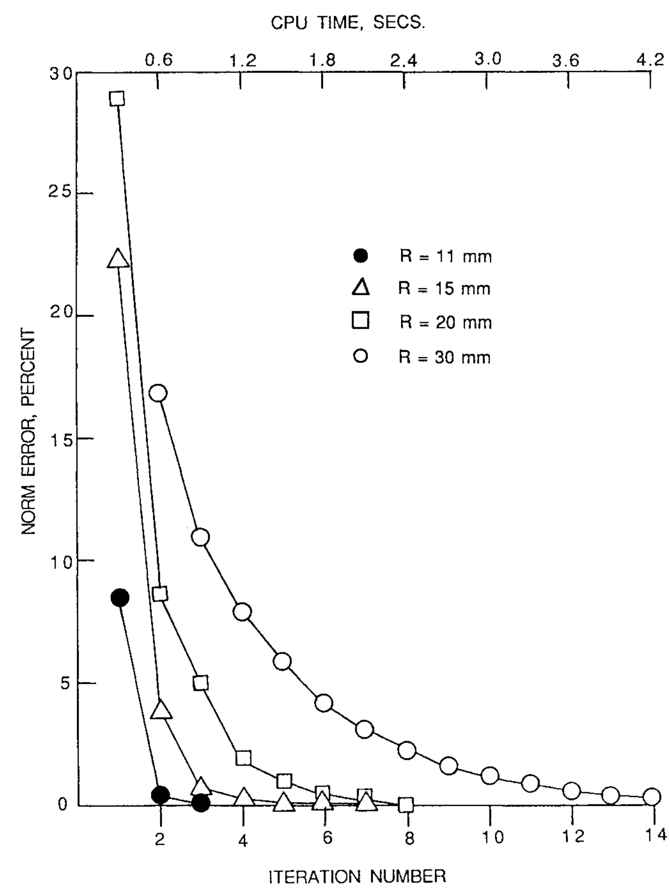

Fig. 5 Convergence of the total iteration algorithm with $\{u_2^o\}_{,L}=\{u_1\}_{,L}$ for the reanalysis of design sensitivities.

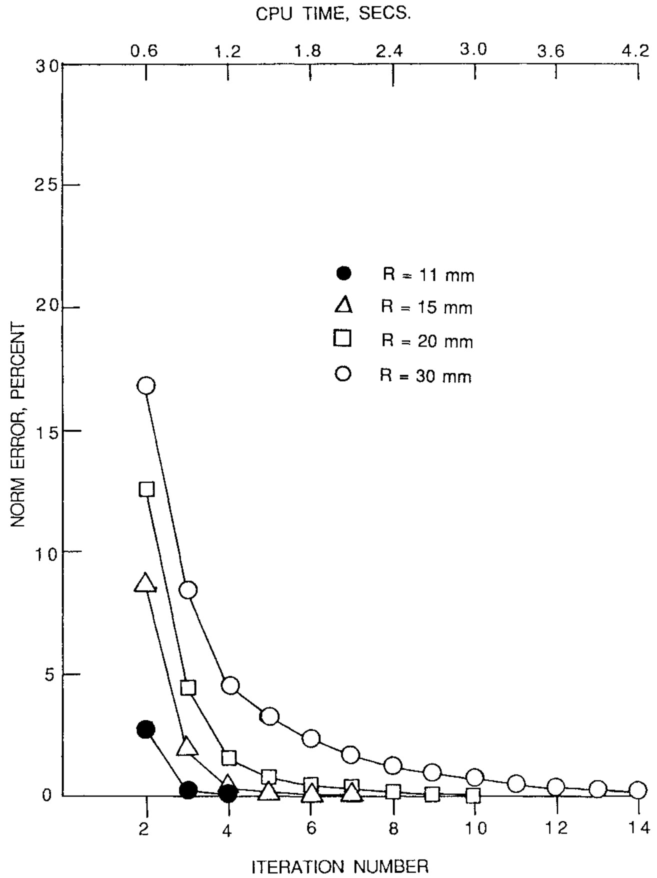

Fig. 6 Convergence of the total iteration algorithm with $\{u_2^0\},_L = \{O\}$ for the reanalysis of design sensitivities.

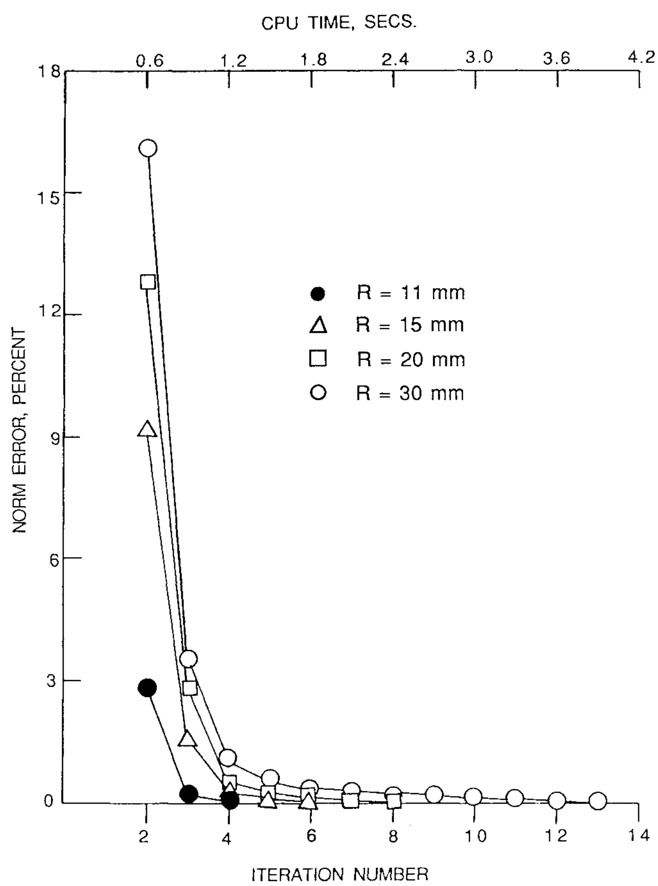

Fig. 7 Convergence of the incremental iteration algorithm for the reanalysis of design sensitivities.

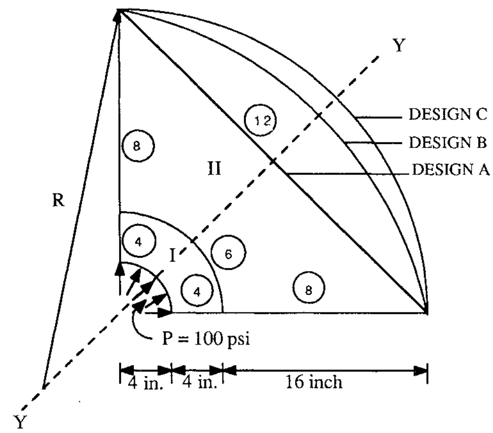

Fig. 8 A two-zone boundary-element model for a thick-walled pipe under internal pressure.

For the present numerical study, the radius R of the circular arc PQ is considered as the design variable. The center O of the arc PQ can lie anywhere along the line YY for the purpose of modification of the geometry of the hole. Four cases corresponding to $R = 30$mm, $R = 20$mm, $R = 15$mm, and $R = 11$mm, respectively, were studied. The objective was to obtain the static response and design sensitivities of the final configuration corresponding to $R = 10$mm (shown as Design C in Fig. 1) assuming each of the above cases, in turn, to be the initial design configuration for the hole.

The quarter-symmetry model of the square shown in Fig. 1 was discretized using 40 quadratic continuous boundary elements. Each of straight edges was modeled using 10 elements and the circular portion PQ was modeled using 20 elements.

After the application of appropriate boundary conditions, a matrix equation with 164 unknown variables was obtained. The results for the static response and the design sensitivities for the modified structure were obtained using both the total iteration algorithm and the incremental iteration algorithm. Within the total iteration algorithm two separate cases were studied: in the first case, the iterations start with the response quantities of the initial design, and in the second case, the iterations start from zero. The convergence of these reanalysis algorithms for the static response and the design sensitivities were shown in Figs. 2–7. It is noted that the case corresponding to $R = 11$mm constitute a realistic modification in design. For this case, the convergence is very rapid as seen in all these figures and only 2 iterations appear to provide adequate accuracy. The other cases shown in these figures constitute a severe modification of these designs. A fast convergence was also observed for these reanalyses.

A direct evaluation of the response variables for the new design would involve the factorization of the $[F_2]$ matrix of size (164×164) for this case. This requires a CPU time of 5 s for the present case. Each iteration step, however, involves only a forward and backward substitution, and requires a CPU time of 0.3 s only. Thus, for the case of $R = 11$mm, while the complete direct analysis requires 5 CPU s, the reanalysis algorithms would require only 0.6 s. This substantial amount of savings using the reanalysis algorithms even for the present example of small dimensions is very encouraging. A significant economy in CPU time for larger problems can be thus obtained using the present method.

Reanalysis for a Multizone Model

A thick-walled pipe under internal pressure was next studied. This example with different dimensions was studied earlier by Wu[16] in which starting from the initial configuration (Design A in Fig. 8) the final optimal configuration of a circle (Design C in Fig. 8) was obtained. In the present study, an intermediate configuration with $R = 36$ inch (Design B in Fig. 8) is assumed as the initial design and the final configuration

Table 1 Comparison of convergence of static response analysis for condensed and uncondensed multizone models

	Static response error norm							
	$\epsilon \leq 0.1$				$\epsilon \leq 0.001\%$			
	Uncondensed		Condensed		Uncondensed		Condensed	
Iteration algorithm	No. of iterations	CPU time, s	No. of iterations	CPU time, s	No. of iterations	CPU time, s	No. of iterations	CPU time, s
Total	4	2.0	2	0.05	6	3.0	3	0.08
Incremental	6	3.0	3	0.08	6	3.0	4	0.1

Table 2 Comparison of convergence of design sensitivity analysis for condensed and uncondensed multizone models

	Design sensitivity error norm							
	$\epsilon \leq 0.1$				$\epsilon \leq 0.001\%$			
	Uncondensed		Condensed		Uncondensed		Condensed	
Iteration algorithm	No. of iterations	CPU time, s	No. of iterations	CPU time, s	No. of iterations	CPU time, s	No. of iterations	CPU time, s
Total	3	1.5	2	0.05	5	2.5	3	0.08
Incremental	5	2.5	2	0.05	6	3.0	3	0.08

(Design C in Fig. 8) is taken to be the modified shape to study the performance of the reanalysis algorithms.

The quarter-symmetry model of the thick-walled pipe shown in Fig. 8 was modeled using two zones and a total of 42 quadratic continuous boundary elements. The element distribution is shown in Fig. 8 by numbers in circles. The radius R with its center lying on the line YY was chosen as the design variable. Two cases of this proglem were studied: for the first case, both zones were left uncondensed resulting in a system consisting of 216 unknown variables, and for the second case both zones were condensed to retain the degrees of freedom along the zone interface resulting in a system consisting of 52 unknown variables. It is noted that the zone interface which contains the master degrees of freedom was chosen closer to the inner wall of the pipe to have a smaller length. This interface was thus modeled using fewer elements resulting in a system of equations with fewer unknowns for which the reanalysis iterations are done.

The iterations required and the corresponding CPU timings for the static response and the design sensitivities were given in Tables 1 and 2, respectively. For both types of analyses, the results were shown corresponding to the total iteration and incremental iteration algorithms. The CPU times and the iterations required for the two-zone uncondensed and condensed models, respectively, were compared in Tables 1 and 2. It is noted that a direct evaluation of solution variables would have required 9.8 CPU s for the uncondensed model and 0.2 CPU s for the condensed model. A considerable saving in CPU time over the direct evaluation was thus obtained using the present reanalysis algorithms. It was also seen from Tables 1 and 2 that it is advantageous to employ the condensation techniques in conjunction with the present reanalysis techniques to achieve economy of CPU times.

Conclusions

This paper presents a total iteration algorithm and an incremental iteration algorithm for the efficient reanalysis of static responses and design sensitivities of modified objects using the boundary-element method. It is demonstrated that the boundary-element system equations are amenable to reanalysis techniques. The factorization of the system coefficient matrix of an object is a computationally expensive step and is avoided for the modified object in the procedure developed. A series of backward and forward substitutions using the factorized-system coefficient matrix of the original structure are used instead. Numerical examples along with CPU times are provided to show the savings obtained using the present developments. The additional advantages of using multizone boundary-element discretization with zone condensation in conjunction with the reanalysis algorithms are also demonstrated. Numerical data is given to show that the iteration algorithms converge rapidly. The present developments provide a valuable tool in the shape-optimization procedure where a design is continuously modified until an optimum shape is obtained. These developments will provide an economic solution for each of the series of modified configurations leading to a substantial saving in the shape optimization of large systems.

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